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Vertical differentiation, network externalities and compatibility decisions: an alternative approach

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Abstract

We characterize the equilibrium of a game in a vertically differentiated market which exhibits network externalities. There are two firms, an incumbent and a potential entrant. Compatibility means in our model that the inherent qualities of the goods are close enough. By choosing its quality, the entrant chooses in the same time to be compatible or not. The maximal quality difference that allows compatibility i.e the compatibility interval is chosen by the incumbent which involves costs increasing with the width of that interval.

We show that in order to have two active firms at price equilibrium, the sufficient condition on the market size of a standard vertical differentiation model remains valid under compatibility. However, an additional condition on the firms' qualities is needed under incompatibility. For a small quality segment, the incumbent can block entry choosing an empty compatibility interval.

At the subgame perfect equilibrium, incompatibility prevails if the quality segment is large and the compatibility costs are high. Compatibility prevails for sufficiently large quality segments and low costs of compatibility. Finally there is no entry if the quality segment is small and the compatibility costs are high.

Keywords: Vertical Differentiation, Compatibility, Network Externalities.
JEL classification: L13, L15, D43.

1 Introduction

Direct network externalities exist when the consumers' utility increases with the number of consumers who buy the same good or compatible goods. Examples of such goods are numerous: phones, fax machines and communication technologies in general. In this paper, we characterize the equilibrium in a vertically differentiated market which

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exhibits direct network externalities. There are two firms, an incumbent and a potential entrant. Compatibility means in our model that the inherent qualities of the goods are close enough. By choosing its quality, the entrant chooses in the same time to be compatible or not. The maximal quality difference that allows compatibility is chosen by the incumbent.

The assumption that products should be close to be compatible seems to be reasonable. For instance, some degrees of similarities between communication protocols are necessary to achieve interconnection (compatibility) between communication networks. Mobiles exhibit nowadays numerous functionalities. From a mobile, we can for example send ringings or photos or video sequences to other mobiles that offer the same functionalities, i.e. that are compatible with our mobile, but we can not do so with old mobiles that do not offer these services and are thus incompatible with our mobile. The incumbent being the first in the market and enjoying a greater technological knowledge, he can make compatibility with its product more or less easy. It is this idea that we try to capture through this model.

We focus in this paper on the equilibrium of a game where an incumbent producing the maximal quality chooses in the first step the maximal quality difference between goods that allows compatibility or what we will call the compatibility interval. This choice involves a cost increasing with the width of that interval. A potential entrant chooses her quality in the second step choosing in the same time to be compatible or not. The entrant has thus two strategies: be compatible with the incumbent and enjoy the benefits from accessing to a large network but renounce to some product differentiation with the incumbent or choose incompatibility, the products are thus more differentiated, price competition is relaxed and the entrant has a market power but a smaller network than under compatibility. After the entrant's quality choice, both firms set prices simultaneously and finally each consumer chooses which good to buy. We completely characterize the subgame perfect equilibrium.

We show that in order to have two active firms at price equilibrium, the sufficient condition on the market size of a standard vertical differentiation model remains valid under compatibility. However an additional condition on the firms' qualities is needed under incompatibility. For a small quality segment, the incumbent has the possibility of deterring entry, which was not possible in the absence of network effects.

We prove that at equilibrium the incumbent may adopt two strategies. Either she chooses a compatibility interval that is the same as the quality segment. This means that whatever the quality choice of the entrant, she is compatible with the incumbent. Or the incumbent chooses an empty compatibility interval. She thus blocks entry under compatibility and this strategy allows her to block completely entry if the quality segment is not large enough. The incumbent's decision depends on the length of the quality segment and the compatibility costs.

We identify three regions in the space (quality segment length - marginal compatibility cost) corresponding to different types of equilibria. There is no entry when the quality segment is very small as the condition to have two active firms under incompatibility is not satisfied and as compatibility is not possible (the incumbent chooses an empty compatibility interval). There is incompatibility if the quality segment is large and the compatibility cost is high. Finally, there is compatibility for a sufficiently large quality segment and for low values of compatibility cost.

Baake and Boom [2] studied a different model of vertical differentiation where the effect of network externalities increases with the inherent quality of firms. Entry is simultaneous and compatibility is achieved by the agreement of both firms on providing an adapter. Contrary to our model there is no relation between compatibility and product differentiation. Moreover they suppose that the quality decision of the firms precedes their compatibility decision. They prove that the firms always agree on compatibility [1]. Their model does not explain why in some cases we can have incompatibility between firms.
Economides and Flyer [3] and Jonard and Shenk [5] linked differently product differentiation and compatibility. For Economides and Flyer [3], compatibility is achieved through the adhesion to a common standard. The compatible products are then identical in non-network characteristics. They find that at equilibrium coalitions that vary greatly in total sales, profits and prices often emerge even if the products and cost structures are identical across firms. Jonard and Shenk [5] studied a circular model of horizontal differentiation. As in our model, they suppose that the firms are closer in the product space if they are compatible. However, compatibility and incompatibility imply exogenously given locations for both firms. Our model is more realistic as we suppose that qualities may be chosen in all the possible interval and that the compatibility interval is endogenous.

Section 2 describes the model. In section 3, we characterize the demand for each firm. In Section 4, we give the solution of the game. We determine the price equilibrium, the quality choice of the entrant thus the compatibility choice and finally the width of the compatibility interval. We conclude in Section 5.

2 The model

We consider a linear model of vertical differentiation with network externalities. An established firm I produces the quality $q_I = \overline{q}$. An entrant e chooses to produce a quality q_e in the segment $[\underline{q}, \overline{q}]$. The entrant can be compatible or incompatible with the incumbent. To achieve compatibility, the product characteristics of firm e must be close to the product characteristics of firm I. Precisely, we suppose the following:

- if $|q_I - q_e| \leq h$, the firms are compatible.
- if $|q_I - q_e| > h$, the firms are incompatible.

The compatibility interval is then $[\overline{q} - h, \overline{q}]$. h is the compatibility interval width i.e the maximal quality difference between firms' products that guarantees their compati-

[1] The same results hold using our utility function.

bility. If the quality difference between the two products is greater than h, the product characteristics of the two firms are so different that they cannot be compatible. As consumers perceive compatible products as close substitutes, choosing to be compatible is then equivalent for the entrant to renounce to some product differentiation.
Depending on the compatibility configuration, the network of a firm consists of either the firm's own sales (when incompatibility prevails: the two firms have different networks) or the sum of the two firms' sales (when compatibility prevails: the two firms have a common network).
Marginal production cost is set equal to zero for both firms.
Let p_i, $i = I$ or e be the price set by firm i , q_i its quality and y_i its network size.
Consumers are characterized by their intensity of preference for quality θ. They are uniformly distributed on $[\underline{\theta}, \overline{\theta}]$. If a consumer buys a unit of product from firm i, his utility is given by:

$$u_i(\theta) = -p_i + \theta q_i + \omega y_i$$

ω represents the network's "intensity". The larger is ω, the more important is network for consumers.
We seek for a subgame perfect Nash equilibrium of the game described by the following steps:

1. Firm I chooses h, which involves the costs $c(h) = \alpha^2 h$. This cost can be considered as a cost of protection of the incumbent quality $\overline{q}$ or, as technical costs to make her product compatible, which are greater as h is larger since the required efforts to make compatible two products are larger as the difference between their qualities is larger. α^2 will be called marginal compatibility cost.

2. Firm e chooses her quality thus deciding to be compatible or not depending on whether she chooses a quality in the compatibility interval or not.

3. Firms I and e set prices simultaneously.

4. Each consumer decides which quality to buy.

The game is solved by backward induction. We determine first the demand for each firm as a function of p_e, p_I and q_e. Then, we determine the price equilibrium and the quality choice of the entrant, for fixed h. Finally h is calculated.
We suppose that $\overline{\theta} > 2\underline{\theta}$, so that both firms can be active under compatibility[2].
We denote by Π_c^e and Π_c^I the profits of both firms under compatibility and by Π_{inc}^e and Π_{inc}^I their profits under incompatibility.
We denote by $A = \frac{\omega(\overline{\theta}+\underline{\theta})}{\overline{\theta}-2\underline{\theta}}$.
The next section deals with consumers' choice. From the consumers' utility function, we have that the decision of a consumer depends on the decisions of others. A sort of recursivity emerges, which implies as we will prove, under some conditions on prices, a multiplicity of consumers' equilibria. A selection rule **(SA)** between these equilibria will be provided precisely later.

[2]This is the condition to have two active firms at equilibrium in a standard vertical differentiation model. It remains sufficient under compatibility but as we will prove later, activity of both firms requires an additional condition on qualities under incompatibility.

3 Demand characterization

Consumers evaluate products in terms of their prices, their inherent qualities and their network sizes thus the groups in which consumers split.

Under compatibility, both firms have the same network. Consumers evaluate products only in terms of their prices and their inherent qualities. The demand functions are determined in this case as in the case without network effects. However under incompatibility, the consumer's decision depends on the other consumers' decisions. Multiple Nash Equilibria relative to the consumers' choice may emerge[3].

Next we first determine the demand function of each firm under compatibility, then the demand functions under incompatibility.

For a consumer θ, denote by $u_I(\theta)$ and $u_e(\theta)$ respectively the utilities of consumer θ when she buys form I and e. We have

$$u_I(\theta) - u_e(\theta) = -p_I + p_e + \theta(\overline{q} - q_e) + \omega(y_I - y_e).$$

Under compatibility the last term of this equality is null as $y_I = y_e$. The utility difference $u_I(\theta) - u_e(\theta)$ is exactly the same as in the absence of network externalities and the demand functions are easily determined since the utility difference $u_I(\theta) - u_e(\theta)$ is independent of other consumers' quality choice. By Lemma 1, we give the demand functions under compatibility.

Lemma 1 *Under compatibility, the demand function of firm I is*

$$D_c^I = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I < p_e + \underline{\theta}(\overline{q} - q_e) \\ \overline{\theta} - \widehat{\theta} & \text{if} \quad p_e + \underline{\theta}(\overline{q} - q_e) \leq p_I \leq p_e + \overline{\theta}(\overline{q} - q_e) \\ 0 & \text{if} \quad p_I > p_e + \overline{\theta}(\overline{q} - q_e) \end{cases}$$

The demand function of firm e is $D_c^e = \overline{\theta} - \underline{\theta} - D_c^I$.
$\widehat{\theta}$ *is equal to* $\frac{p_I - p_e}{\overline{q} - q_e}$. *It represents the marginal consumer indifferent between* q_e *and* $\overline{q}$.

Under compatibility, the demand functions are the classical demand functions in a vertical differentiation model without network externalities (Anderson et al for instance [1]). As firms have the same network, the network size is no more a criterion for the choice of the quality to buy.

We now study the demand functions under incompatibility. The characterization of the demand in this case is more complex than under compatibility because the utility function of a consumer θ and consequently the quality choice of a consumer depend on the quality choice of other consumers. A sort of recursivity appears. The characterization of the demand when there is incompatibility amounts to the determination of a Nash equilibrium between consumers where at equilibrium no consumer has interest to change his quality choice. We will call the solution of the last step of the game a consumers' Nash equilibrium or a network equilibrium.

In Lemma 2, we prove that three situations may emerge at a consumers' Nash Equilibrium: only firm I is active, only firm e is active, or both firms are active.

[3] As in Baake and Boom [2]

Lemma 2 *At a consumers' Nash equilibrium, if there exists a consumer θ_0 that prefers the quality $\overline{q}$ or that is indifferent between the qualities of both firms, then every consumer θ such that $\theta > \theta_0$ prefers $\overline{q}$.*

Lemma 2 shows that each firm's demand is necessarily an interval (possibly empty) and that the intervals are ordered as in the standard case: the lowest θ buy the lowest quality and the highest θ buy the highest quality.

Proof. For such a θ_0 we have $u_I(\theta_0) - u_e(\theta_0) \geq 0$ which is equivalent to $\theta_0(\overline{q} - q_e) \geq p_I - p_e - \omega(y_I - y_e)$.
Consider now a consumer θ_1 such that $\theta_1 > \theta_0$. This implies $\theta_1(\overline{q} - q_e) > \theta_0(\overline{q} - q_e) \geq p_I - p_e - \omega(y_I - y_e)$ which means that $u_I(\theta_1) - u_e(\theta_1) > 0$ thus consumer θ_1 prefers $\overline{q}$.
■

Note that the same arguments may be used to prove that if at a consumers' Nash equilibrium, a consumer θ_0 prefers the quality q_e then every consumer θ such that $\theta < \theta_0$ prefers q_e.

Lemma 2 also allows to identify three possible types of consumers' Nash equilibria:

1. Only firm I is active: $y_e = 0$ and $y_I = \overline{\theta} - \underline{\theta}$. We call this type of Nash equilibrium a type 1 Nash equilibrium.

2. Only firm e is active: $y_e = \overline{\theta} - \underline{\theta}$ and $y_I = 0$. We call this type of Nash equilibrium a type 2 Nash equilibrium.

3. Both firms have positive sales: by Lemma 2, we have $y_e = \widehat{\theta} - \underline{\theta}$ and $y_I = \overline{\theta} - \widehat{\theta}$. We call this type of Nash equilibrium a type 3 Nash equilibrium. The marginal consumer $\widehat{\theta}$ indifferent between $\overline{q}$ and q_e is necessarily given by $u_I(\widehat{\theta}) = u_e(\widehat{\theta})$, which is equivalent to

$$-p_I + \widehat{\theta}\overline{q} + \omega(\overline{\theta} - \widehat{\theta}) = -p_e + \widehat{\theta}q_e + \omega(\widehat{\theta} - \underline{\theta}),$$

which yields

$$\widehat{\theta} = \frac{p_I - p_e - \omega(\overline{\theta} + \underline{\theta})}{\overline{q} - q_e - 2\omega}$$

if $\overline{q} - q_e - 2\omega \neq 0$. [4]

Denote by

$$\varphi(p_e) = p_e + \underline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta})$$

and

$$\psi(p_e) = p_e + \overline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}).$$

By Lemmas 3, 4 and 5, we determine the conditions under which each type of consumers' Nash equilibria exists.

Lemma 3 *A type 1 consumers' equilibrium (where only firm I is active) exists if and only if $p_I \leq \varphi(p_e)$.*

[4] The case $\overline{q} - q_e - 2\omega = 0$ is examined later.

Proof. Consider some consumer $\theta \in [\underline{\theta}, \overline{\theta}]$.
$u_I(\theta) - u_e(\theta) = -p_I + p_e + \theta(\overline{q} - q_e) + \omega(y_I - y_e)$.
If all the other consumers buy $\overline{q}$ then $y_I = \overline{\theta} - \underline{\theta}$ and $y_e = 0$.
θ also prefers $\overline{q}$ if $-p_I + p_e + \theta(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) > 0$. An equilibrium where only firm I is active exists if

$$-p_I + p_e + \theta(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) > 0 \text{ for every } \theta \in]\underline{\theta}, \overline{\theta}]$$

Consumer $\underline{\theta}$ either prefers $\overline{q}$ or is indifferent between both qualities. We have then

$$-p_I + p_e + \underline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) \geq 0$$

which yields

$$p_I \leq \varphi(p_e).$$

■

Lemma 4 *A type 2 consumers' equilibrium (where only firm e is active) exists if and only if* $p_I \geq \psi(p_e)$.

Proof. If consumer θ supposes that only firm e is active then $y_I = 0$ and $y_e = \overline{\theta} - \underline{\theta}$.
Consumer θ also prefers q_e if $u_I(\theta) - u_e(\theta) = -p_I + p_e + \theta(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) < 0$.
An equilibrium where only firm e is active exists if

$$-p_I + p_e + \theta(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) < 0 \text{ for every } \theta \in [\underline{\theta}, \overline{\theta}[$$

Consumer $\overline{\theta}$ either prefers q_e or is indifferent between both qualities. We have then

$$-p_I + p_e + \overline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) \leq 0$$

which yields

$$p_I \geq \psi(p_e).$$

■

Lemma 5 *A type 3 consumers' equilibrium (where both firms are active) exists if and only if*

- $\varphi(p_e) < p_I < \psi(p_e)$ *when* $\overline{q} - q_e - 2\omega > 0$
- $\psi(p_e) < p_I < \varphi(p_e)$ *when* $\overline{q} - q_e - 2\omega < 0$

Proof. If a consumer θ supposes that both firms are active, then he knows that there exists a marginal consumer $\widehat{\theta} \in]\underline{\theta}, \overline{\theta}[$ indifferent between $\overline{q}$ and q_e (otherwise only one firm is active) and that according to Lemma 2, the market is divided according to the rule:

- consumers in $[\underline{\theta}, \widehat{\theta}]$ buy q_e
- consumers in $[\widehat{\theta}, \overline{\theta}]$ buy $\overline{q}$.

We prove that consumer θ has no interest to deviate from the specified rule. The marginal consumer $\widehat{\theta}$ is such that $u_I(\widehat{\theta}) = u_e(\widehat{\theta})$.
$u_I(\theta) - u_e(\theta) = u_I(\theta) - u_I(\widehat{\theta}) - (u_e(\theta) - u_e(\widehat{\theta})) = (\overline{q} - q_e)(\theta - \widehat{\theta})$. Thus $u_I(\theta) - u_e(\theta)$ has the same sign as $\theta - \widehat{\theta}$ and consumer θ behaves according to the rule.
An equilibrium with two active firms exists if and only if $\underline{\theta} < \widehat{\theta} < \overline{\theta}$ which is equivalent to the conditions cited in the lemma. ■

We are now ready to characterize the firms' demands under incompatibility.

Proposition 1 *Under incompatibility, if $\overline{q} - q_e - 2\omega > 0$ then the demand function of firm I is given by:*

$$D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I \leq \varphi(p_e) \\ \overline{\theta} - \widehat{\theta} & \text{if} \quad \varphi(p_e) < p_I < \psi(p_e) \\ 0 & \text{if} \quad p_I \geq \psi(p_e) \end{cases}$$

where $\widehat{\theta} = \frac{p_I - p_e - \omega(\overline{\theta} + \underline{\theta})}{\overline{q} - q_e - 2\omega}$.

When $\overline{q} - q_e - 2\omega > 0$ there is no indetermination concerning the consumers' behavior. For each price, only one consumers' Nash equilibrium prevails.

Proof. When $\overline{q} - q_e - 2\omega > 0$, $\varphi(p_e) < \psi(p_e)$. Thus from Lemmas 3, 4 and 5, in each type of interval ($p_I \in [0, \varphi(p_e)]$ or $p_I \in]\varphi(p_e), \psi(p_e)[$ or $p_I \in [\psi(p_e), y]$), there exists a unique consumers' Nash equilibrium. ■

Proposition 2 *Under incompatibility, if $\overline{q} - q_e - 2\omega < 0$ then, depending on the selected consumers' Nash equilibrium in the interval $]\psi(p_e), \varphi(p_e)[$, the demand function of firm I is given by:*

- $D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I \leq \varphi(p_e) \\ 0 & \text{if} \quad p_I > \varphi(p_e) \end{cases}$ *if consumers select a type 1 NE.*
- $D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I < \psi(p_e) \\ 0 & \text{if} \quad p_I \geq \psi(p_e) \end{cases}$ *if consumers select a type 2 NE.*
- $D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I < \psi(p_e) \\ \overline{\theta} - \widehat{\theta} & \text{if} \quad \psi(p_e) \leq p_I \leq \varphi(p_e) \\ 0 & \text{if} \quad p_I > \varphi(p_e) \end{cases}$ *if consumers select a type 3 NE.*

where $\widehat{\theta} = \frac{p_I - p_e - \omega(\overline{\theta} + \underline{\theta})}{\overline{q} - q_e - 2\omega}$.

Proof. When $\overline{q} - q_e - 2\omega < 0$, $\psi(p_e) < \varphi(p_e)$ so:

- if $p_I < \psi(p_e)$ then $p_I < \varphi(p_e)$ and only an equilibrium of type 1 exists.
- if $p_I > \varphi(p_e)$ then $p_I > \psi(p_e)$ and only an equilibrium of type 2 exits.
- if $\psi(p_e) < p_I < \varphi(p_e)$ then the three types of equilibrium can exist as the price p_I simultaneously satisfies the conditions of Lemmas 3, 4 and 5.
- if $p_I = \psi(p_e)$ then consumer $\overline{\theta}$ is indifferent between $\overline{q}$ and q_e and all the other consumers prefer q_e. Only firm e is then active.
- if $p_I = \varphi(p_e)$ then consumer $\underline{\theta}$ is indifferent between $\overline{q}$ and q_e and all the other consumers prefer $\overline{q}$. Only firm I is then active.

■

We now examine the special case $\overline{q} - q_e - 2\omega = 0$.

Proposition 3 *Under incompatibility, if $\overline{q} - q_e - 2\omega = 0$ then, depending on the selected consumers' Nash equilibrium, the demand function of firm I is given by:*

$$D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I \leq p_e + \omega(\overline{\theta} + \underline{\theta}) \\ 0 & \text{if} \quad p_I > p_e + \omega(\overline{\theta} + \underline{\theta}) \end{cases} \quad \textit{if consumers select a type 1 NE.}$$

$$D^I_{inc} = \begin{cases} \overline{\theta} - \underline{\theta} & \text{if} \quad p_I < p_e + \omega(\overline{\theta} + \underline{\theta}) \\ 0 & \text{if} \quad p_I \geq p_e + \omega(\overline{\theta} + \underline{\theta}) \end{cases} \quad \textit{if consumers select a type 2 NE.}$$

Proof. When $\overline{q} - q_e - 2\omega = 0$, $\psi(p_e) = \varphi(p_e) = p_e + \omega(\overline{\theta} + \underline{\theta})$ thus:

- If $p_I < p_e + \omega(\overline{\theta} + \underline{\theta})$ then only an equilibrium of type 1 exists.
- If $p_I > p_e + \omega(\overline{\theta} + \underline{\theta})$ then only an equilibrium of type 2 exists.
- If $p_I = p_e + \omega(\overline{\theta} + \underline{\theta})$ then both type 1 and 2 equilibria can exist as the price p_I satisfies simultaneously the conditions of Lemmas 3 and 4.

■

As multiple consumers' Nash equilibria can exist under incompatibility when $\overline{q} - q_e - 2\omega \leq 0$, we specify by the next assumption a selection rule between consumers' Nash equilibria.

Assumption (SA) *Consumers always choose a type 1 Nash equilibrium when multiple consumers' Nash equilibria exist.*

Assumption **(SA)** means that, if a selection is needed, consumers select the equilibrium where only firm I is active. Indeed a consumers' Nash equilibrium of type 2 is possible if each consumer believes that all the other consumers will choose the quality q_e. A type 3 consumers' Nash equilibrium is possible if consumers believe that the other consumers are shared between both firms in good order (Consumers between $\underline{\theta}$ and $\hat{\theta}$ buy from e while the others buy from I). As the incumbent is the first in the market, consumers are *a priori* accustomed to buy from the incumbent. It would thus be difficult to imagine that all consumers or some of them (well situated) change their minds in the same time to buy from the entrant. As the selection of an equilibrium amounts to a coordination problem, the coordination is easier to imagine with a type 1 consumers' Nash equilibrium.

Moreover when consumers choose a type 2 Nash equilibrium, we show that there may be an existence problem of a price equilibrium and thus an existence problem of a game equilibrium (Appendix 1). The selection of a type 3 Nash equilibrium by consumers is studied in Appendix 2. We show that a price equilibrium never exists in this case.

4 The solution of the game

We now determine the price equilibrium, the entrant's quality choice and the incumbent's choice of h given assumption **(SA)**.

4.1 Price equilibrium

We solve the step of the game corresponding to the choice of prices for given h and q_e. Proposition 4 provides the price equilibrium under compatibility ($\overline{q}-q_e \leq h$). Lemmas 6, 7 and 8 provide the price equilibrium under incompatibility depending on the quality difference between the firms'products. Proposition 5 summarizes these lemmas giving the price equilibrium in all cases under incompatibility.

Proposition 4 *Under compatibility, the price equilibrium of the game is charaterized by:*

$$\begin{cases} p_c^I = \frac{2\overline{\theta}-\underline{\theta}}{3}(\overline{q}-q_e) \\ p_c^e = \frac{\overline{\theta}-2\underline{\theta}}{3}(\overline{q}-q_e) \end{cases} \text{ with the profits } \begin{cases} \Pi_c^I = (\frac{2\overline{\theta}-\underline{\theta}}{3})^2(\overline{q}-q_e) \\ \Pi_c^e = (\frac{\overline{\theta}-2\underline{\theta}}{3})^2(\overline{q}-q_e) \end{cases}.$$

Proof. Under compatibility, the effect of the network size disappears as both firms have the same network. To make their choice, consumers consider only qualities and prices as in a classical vertical differentiation model. Proposition 4 is provided only for completeness. Its proof can be found for instance in Anderson et al. [1]. ■

We now deal with the price equilibrium under incompatibility in the second step of the game. The three cases that emerged in the characterization of the demand have naturally to be distinguished here.
Recall that $A = \frac{\omega(\overline{\theta}+\underline{\theta})}{\overline{\theta}-2\underline{\theta}}$.

Lemma 6 *Under incompatibility, if $\overline{q}-q_e-2\omega > 0$, the equilibrium prices and profits are given by:*

- *if* $\overline{q}-q_e-2\omega > A$
$\begin{cases} p^I_{inc} = \frac{2\overline{\theta}-\underline{\theta}}{3}(\overline{q}-q_e-2\omega)+\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \\ p^e_{inc} = \frac{\overline{\theta}-2\underline{\theta}}{3}(\overline{q}-q_e-2\omega)-\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \end{cases}$ $\begin{cases} \Pi^I_{inc} = (\frac{2\overline{\theta}-\underline{\theta}}{3}+\frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) \\ \Pi^e_{inc} = (\frac{\overline{\theta}-2\underline{\theta}}{3}-\frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) \end{cases}$
- *if* $0 < \overline{q}-q_e-2\omega \leq A$
$\begin{cases} p^I_{inc} = \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ p^e_{inc} = 0 \end{cases}$ $\begin{cases} \Pi^I_{inc} = (\underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}))(\overline{\theta}-\underline{\theta}) \\ \Pi^e_{inc} = 0 \end{cases}$

Proof. By Proposition 1, we have that firm I's profit is given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_I < \varphi(p_e) \\ p_I(\overline{\theta}-\widehat{\theta}) & \text{if} \quad \varphi(p_e) \leq p_I \leq \psi(p_e) \\ 0 & \text{if} \quad p_I > \psi(p_e) \end{cases}$$

Firm e's profit is given by:

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_e \leq p_I-\underline{\theta}(\overline{q}-q_e)-\omega(\overline{\theta}-\underline{\theta}) \\ p_e(\widehat{\theta}-\underline{\theta}) & \text{if} \quad p_I-\underline{\theta}(\overline{q}-q_e)-\omega(\overline{\theta}-\underline{\theta}) < p_e \leq p_I-\overline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ 0 & \text{if} \quad p_e > p_I-\overline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \end{cases}$$

The best reply correspondences are thus given by:

$$R^I(p_e) = \begin{cases} \frac{p_e+\overline{\theta}(\overline{q}-q_e)-\omega(\overline{\theta}-\underline{\theta})}{2} & \text{if} \quad p_e < (\overline{\theta}-2\underline{\theta})(\overline{q}-q_e-2\omega-A) \\ p_e+\underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_e \geq (\overline{\theta}-2\underline{\theta})(\overline{q}-q_e-2\omega-A) \end{cases}$$

$$R^e(p_I) = \begin{cases} p_I-\underline{\theta}(\overline{q}-q_e)-\omega(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_I \geq \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ [0,y] & \text{if} \quad p_I < \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \end{cases}$$

The best reply correspondences intersect at

$$\begin{cases} p_I = \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ p_e = 0 \end{cases} \text{if } \overline{q}-q_e-2\omega \leq A$$

Otherwise the best reply correspondences intersect at

$$\begin{cases} p_I = \frac{2\overline{\theta}-\underline{\theta}}{3}(\overline{q}-q_e-2\omega)+\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \\ p_e = \frac{\overline{\theta}-2\underline{\theta}}{3}(\overline{q}-q_e-2\omega)-\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \end{cases}$$

■

In the absence of network externalities, the condition $\overline{\theta} > 2\underline{\theta}$ ensures that the market is large enough to have at least two active firms (see for example Anderson et al [1]). When network externalities are introduced, under compatibility this condition remains sufficient to have at least two active firms. However, under incompatibility an additional condition is needed which is $\overline{q}-q_e-2\omega > A$, which means that the quality segment must be sufficiently large to allow the activity of two firms.

We now examine the price equilibrium when $\overline{q}-q_e-2\omega < 0$.

Lemma 7 *Under incompatibility, if $\overline{q}-q_e-2\omega < 0$ and under assumption* **(SA)**, *the equilibrium prices and profits are given by:*

$$\begin{cases} p^I_{inc} = \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ p^e_{inc} = 0 \end{cases} \begin{cases} \Pi^I_{inc} = (\underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}))(\overline{\theta}-\underline{\theta}) \\ \Pi^e_{inc} = 0 \end{cases}.$$

Proof. By Proposition 2, we have that the profit of firms I and e are given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_I \leq \varphi(p_e) \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_e < p_I - \underline{\theta}(\overline{q}-q_e) - \omega(\overline{\theta}-\underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

The best reply correspondence of firm I is then $R_I(p_e) = \varphi(p_e) = p_e + \underline{\theta}(\overline{q}-q_e) + \omega(\overline{\theta}-\underline{\theta})$ for every $p_e \in [0, y]$. The best reply correspondence of firm e is given by:

$$R^e(p_I) = \begin{cases} \text{doesn't exist} & \text{if} \quad p_I > \underline{\theta}(\overline{q}-q_e) + \omega(\overline{\theta}-\underline{\theta}) \\ [0,y] & \text{if} \quad p_I \leq \underline{\theta}(\overline{q}-q_e) + \omega(\overline{\theta}-\underline{\theta}) \end{cases}$$

The unique possible intersection between the best reply correspondences is $(p_I = \underline{\theta}(\overline{q}-q_e) + \omega(\overline{\theta}-\underline{\theta}), p_e = 0)$. ■

Lemma 8 *Under incompatibility, if $\overline{q}-q_e-2\omega = 0$ and under assumption* **(SA)***, the equilibrium prices and profits are given by:*

$$\begin{cases} p^I_{inc} = \omega(\overline{\theta}+\underline{\theta}) \\ p^e_{inc} = 0 \end{cases} \quad \begin{cases} \Pi^I_{inc} = \omega(\overline{\theta}^2 - \underline{\theta}^2) \\ \Pi^e_{inc} = 0 \end{cases}$$

Note that the prices are exactly the same as in the cases $\overline{q}-q_e-2\omega < 0$ and $\overline{q}-q_e-2\omega > 0$ replacing $\overline{q}-q_e$ by 2ω. The case $\overline{q}-\underline{q}-2\omega = 0$ is thus obtained as a limit of these cases and need not be dealt with separately from now on.

Proof. By Proposition 3, we have that the profits of firms I and e are given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_I \leq p_e + \omega(\overline{\theta}+\underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta}-\underline{\theta}) & \text{if} \quad p_e < p_I - \omega(\overline{\theta}+\underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

The best reply correspondence of firm I is then $R_I(p_e) = p_e + \omega(\overline{\theta}+\underline{\theta})$ for every $p_e \in [0, y]$. The best reply correspondence of firm e is given by:

$$R^e(p_I) = \begin{cases} \text{doesn't exist} & \text{if} \quad p_I > \omega(\overline{\theta}+\underline{\theta}) \\ [0,y] & \text{if} \quad p_I \leq \omega(\overline{\theta}+\underline{\theta}) \end{cases}$$

The unique possible intersection between the best reply correspondences is $(p_I = \omega(\overline{\theta}+\underline{\theta}), p_e = 0)$. ■

Using assumption **(SA)** when needed, two situations may emerge at price equilibrium. When $\overline{q}-q_e > A+2\omega$, both firms are active. When $\overline{q}-q_e \leq A+2\omega$, only firm I is active. Proposition 5 summarizes all the lemmas providing the price equilibrium in all cases.

Proposition 5 *Under incompatibility, using assumption* **(SA)** *when needed, the price equilibrium of the game is characterized by:*

- *If* $\overline{q}-q_e > A+2\omega$ *then both firms are active at price equilibrium. The equilibrium prices and profits are given by*
$\begin{cases} p^I_{inc} = \frac{2\overline{\theta}-\underline{\theta}}{3}(\overline{q}-q_e-2\omega)+\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \\ p^e_{inc} = \frac{\overline{\theta}-2\underline{\theta}}{3}(\overline{q}-q_e-2\omega)-\frac{\omega}{3}(\underline{\theta}+\overline{\theta}) \end{cases}$ $\begin{cases} \Pi^I_{inc} = (\frac{2\overline{\theta}-\underline{\theta}}{3}+\frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) \\ \Pi^e_{inc} = (\frac{\overline{\theta}-2\underline{\theta}}{3}-\frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) \end{cases}$

- *If* $\overline{q}-q_e \leq A+2\omega$ *then only firm I is active at price equilibrium. The equilibrium prices and profits are given by*
$\begin{cases} p^I_{inc} = \underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}) \\ p^e_{inc} = 0 \end{cases}$ $\begin{cases} \Pi^I_{inc} = (\underline{\theta}(\overline{q}-q_e)+\omega(\overline{\theta}-\underline{\theta}))(\overline{\theta}-\underline{\theta}) \\ \Pi^e_{inc} = 0 \end{cases}$

4.2 The entrant's quality decision

We determine in this section the quality choice of the entrant which corresponds to the solution of the second step of the game. For a given h, the entrant must choose either to be compatible, i.e $q_e \in [\overline{q}-h, \overline{q}]$ or to be incompatible, i.e $q_e \in [\underline{q}, \overline{q}-h[$. According to Proposition 5, the resulting price equilibrium under incompatibility depends on the position of $\overline{q}-q_e$ relative to $A+2\omega$. A discussion on h, $A+2\omega$ and $\overline{q}-\underline{q}$ is thus needed. The obvious case $h = \overline{q}-\underline{q}$ is dealt with in Lemma 9. Then, we examine the quality equilibrium when $h \in [0, \overline{q}-\underline{q}[$. Two cases are distinguished: (Lemmas 10 and 11)

- $\overline{q}-\underline{q} > A+2\omega$: for $q_e \in [\underline{q}, \min(\overline{q}-A-2\omega, \overline{q}-h)[$, both firms can be active under incompatibility.
- $\overline{q}-\underline{q} \leq A+2\omega$: for every $q_e \in [\underline{q}, \overline{q}-h[$, only firm I can be active under incompatibility.

Lemma 9 *If* $h = \overline{q}-\underline{q}$ *then the entrant chooses* $\underline{q}$ *and compatibility holds at equilibrium.*

Proof. If $h = \overline{q}-\underline{q}$ then for every $q_e \in [\underline{q}, \overline{q}]$ we have $\overline{q}-q_e \leq h$. Only compatibility is then possible and the entrant chooses $\underline{q}$ as her profit under compatibility given by Proposition 4 is decreasing in q_e. ■

Choosing to be incompatible amounts to choose q_e such that $\overline{q}-q_e > h$. $h \geq A+2\omega$ is possible when $\overline{q}-\underline{q} > A+2\omega$ and implies that for every q_e such that $\overline{q}-q_e > h$, we have $\overline{q}-q_e > A+2\omega$, which implies according to Proposition 5 that both firms are active under incompatibility. Lemma 10 results from this discussion.

Lemma 10 *If* $\overline{q}-\underline{q} > A+2\omega$ *then at quality equilibrium, the entrant's quality choice is either* $\underline{q}$ *or* $\overline{q}-h$.

Proof. Suppose $\overline{q}-\underline{q} > A+2\omega$. Choosing to be incompatible amounts to choose q_e such that $\overline{q}-q_e > h$.

- if $h \in [A+2\omega, \overline{q}-\underline{q}[$ (which is a non empty interval since $\overline{q}-\underline{q} > A+2\omega$), for all q_e satisfying $\overline{q}-q_e > h$ we have $\overline{q}-q_e > A+2\omega$. According to Proposition 5, the entrant's profit is given by:
$$\Pi^e(q_e) = \begin{cases} (\frac{\overline{\theta}-2\underline{\theta}}{3} - \frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) & \text{if} \quad \underline{q} \le q_e < \overline{q}-h \\ (\frac{\overline{\theta}-2\underline{\theta}}{3})^2(\overline{q}-q_e) & \text{if} \quad \overline{q}-h \le q_e \le \overline{q} \end{cases}$$
The profit is discontinuous at $\overline{q}-h$. From simple calculations of the derivative, we show that $\Pi^e(q_e)$ is decreasing on $[\underline{q}, \overline{q}-h[$ and decreasing on $[\overline{q}-h, \overline{q}]$. The entrant's profit maximization problem admits a solution ($\underline{q}$ or $\overline{q}-h$).

- if $h \in [0, A+2\omega[$, we have $\underline{q} < \overline{q}-A-2\omega < \overline{q}-h \le \overline{q}$. Two cases have to be distinguished for q_e such that $\overline{q}-q_e > h$: $\underline{q} \le q_e < \overline{q}-A-2\omega$ i.e $\overline{q}-q_e > A+2\omega$ and $\overline{q}-A-2\omega \le q_e < \overline{q}-h$ i.e $A+2\omega \ge \overline{q}-q_e > h$. Replacing in each interval the prices by their values at equilibrium according to Proposition 5, we obtain:
$$\Pi^e(q_e) = \begin{cases} (\frac{\overline{\theta}-2\underline{\theta}}{3} - \frac{\omega(\underline{\theta}+\overline{\theta})}{3(\overline{q}-q_e-2\omega)})^2(\overline{q}-q_e-2\omega) & \text{if} \quad \underline{q} \le q_e < \overline{q}-A-2\omega \\ 0 & \text{if} \quad \overline{q}-A-2\omega \le q_e < \overline{q}-h \\ (\frac{\overline{\theta}-2\underline{\theta}}{3})^2(\overline{q}-q_e) & \text{if} \quad \overline{q}-h \le q_e \le \overline{q} \end{cases}$$
The profit is discontinuous at $\overline{q}-h$. It is decreasing on $[\underline{q}, \overline{q}-2\omega-A]$, null on $[\overline{q}-2\omega-A, \overline{q}-h[$ and decreasing on $[\overline{q}-h, \overline{q}]$. The entrant's profit maximization problem admits a solution ($\underline{q}$ or $\overline{q}-h$).

■

In the next corollary, we determine the optimal quality choice of the entrant when $\overline{q}-\underline{q} > A+2\omega$.
Denote by $x_h^* = A + \frac{h+\sqrt{h(4A+h)}}{2}$. We have $x_h^* > A$ for $h > 0$ and $x_{h=0}^* = A$. We easily deduce from Lemma 10 the following corollary.

Corollary 1 *For $h \in [0, \overline{q}-\underline{q}[$, if $\overline{q}-\underline{q} > A+2\omega$ the entrant chooses:*

- *$\underline{q}$ and thus incompatibility if $\overline{q}-\underline{q} > x_h^* + 2\omega$.*
- *$\overline{q}-h$ and thus compatibility if $A+2\omega < \overline{q}-\underline{q} \le x_h^* + 2\omega$.*

Proof. By Lemma 10, the entrant chooses either $\underline{q}$ or $\overline{q}-h$.
If $h = 0$, the entrant's profit is null at $\overline{q}$ and the optimal choice of the entrant is $\underline{q}$ (the entrant's profit is strictly positive in this case).
If $h > 0$, we have:
$\frac{\Pi^e(\underline{q})}{\Pi^e(\overline{q}-h)} = \frac{\overline{q}-\underline{q}-2\omega}{h}(1 - \frac{\omega(\underline{\theta}+\overline{\theta})}{(\overline{\theta}-2\underline{\theta})(\overline{q}-\underline{q}-2\omega)})^2$

Let $f(x) = \frac{x}{h}(1-\frac{A}{x})^2$ with $A = \frac{\omega(\underline{\theta}+\overline{\theta})}{\overline{\theta}-2\underline{\theta}}$ and $x = \overline{q}-\underline{q}-2\omega$.
$f'(x) = \frac{1}{h}(1-\frac{A^2}{x^2}) > 0$

x	A		$+\infty$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	$+\infty$

The variation table shows that there exists a unique $x_h^* \in]A, +\infty[$ such that $f(x_h^*) = 1$. Solving the equation $f(x) = 1$ yields $x_h^* = \frac{2A+h+\sqrt{h(4A+h)}}{2}$ (the second root of the equation is smaller than A).
Hence if $x > x_h^*$ then the entrant chooses $\underline{q}$ thus incompatibility and if $x \leq x_h^*$ then the entrant chooses $\overline{q} - h$ thus compatibility. ■

Now if $\overline{q} - \underline{q} \leq A + 2\omega$ then for every q_e in $[\underline{q}, \overline{q} - h[$ we have $\overline{q} - q_e < A + 2\omega$. Only firm I is then active under incompatibility. This is detailed in Lemma 11.

Lemma 11 *If $\overline{q} - \underline{q} \leq A + 2\omega$ then at quality equilibrium, we have:*

- *the entrant chooses $\overline{q} - h$ thus compatibility if $h > 0$.*
- *there is no entry if $h = 0$.*

Hence, choosing $h = 0$ when $\overline{q} - \underline{q} \leq A + 2\omega$ is a way to deter entry. Indeed in this case, on the one hand compatibility generates no profit for the entrant as it authorizes no differentiation between the two firms. On the other hand, under incompatibility the condition $\overline{q} - q_e > A + 2\omega$ needed to guarantee a positive profit for the entrant can never be satisfied.

Proof. If $\overline{q} - \underline{q} \leq A + 2\omega$ then for every $q_e \in [\underline{q}, \overline{q} - h[$ we have $\overline{q} - q_e < A + 2\omega$. Due to Proposition 5, the entrant profit expression is:

$$\Pi^e(q_e) = \begin{cases} 0 & \text{if} \quad \underline{q} \leq q_e < \overline{q} - h \\ (\frac{\overline{\theta} - 2\underline{\theta}}{3})^2(\overline{q} - q_e) & \text{if} \quad \overline{q} - h \leq q_e \leq \overline{q} \end{cases}$$

The maximal value of the entrant's profit is reached at $q_e = \overline{q} - h$ if $h > 0$. If $h = 0$ the entrant's profit is null on $[\underline{q}, \overline{q}]$ and there is no entry. ■

Remark: For $h = 0$, when $\overline{q} - \underline{q} \leq A + 2\omega$, there is an indetermination in the choice of q_e (for every $q_e \in [\underline{q}, \overline{q}]$ the entrant's profit is null) which leads to a similar indetermination in the incumbent's price, as $p_I = \underline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta})$. We adopt the convention that $q_e = \underline{q}$ so that $p_I = \underline{\theta}(\overline{q} - \underline{q}) + \omega(\overline{\theta} - \underline{\theta})$ as it is done in classical vertical differentiation (for instance Anderson et al [1], in the case $\overline{\theta} < 2\underline{\theta}$).

Proposition 6 gives the different strategies of the entrant for a given $h \in [0, \overline{q} - \underline{q}[$.

Proposition 6 *The entrant has three different strategies depending on h and the length of the quality segment:*

- *There is no entry if ($h = 0$ and $\overline{q} - \underline{q} \leq A + 2\omega$).*
- *The entrant is active and chooses $\underline{q}$ thus incompatibility if ($\overline{q} - \underline{q} > x_h^* + 2\omega$).*
- *The entrant is active and chooses $\overline{q} - h$ thus compatibility if ($h > 0$ and $\overline{q} - \underline{q} \leq x_h^* + 2\omega$).*

We deduce from Proposition 6 that the entrant prefers incompatibility in two cases:

- If compatibility is not possible ($h = 0$) and the quality segment is sufficiently large ($\overline{q} - \underline{q} > A + 2\omega$) so that the entrant makes a positive profit under incompatibility [5].
- If the quality segment is sufficiently large with respect to h when $h > 0$ ($\overline{q} - \underline{q} > x_h^* + 2\omega$). In other words, if firm e's gains from having a large market power outweighs firm e's gains from the network effects.

If the quality segment is not large enough ($\overline{q} - \underline{q} \leq x_h^* + 2\omega$) and compatibility is possible ($h > 0$), the entrant prefers compatibility. The gains of firm e from being compatible outweighs the gain from having a large market power.
As the next step is the calculation of h, Proposition 6 can be usefully written differently (in terms of h) as follows:

Proposition 6' *The optimal entrant's strategy is:*

- *No entry if ($\overline{q} - \underline{q} \leq A + 2\omega$ and $h = 0$).*
- *$\underline{q}$ thus incompatibility if ($\overline{q} - \underline{q} > A + 2\omega$ and $h < \frac{(\overline{q}-\underline{q}-2\omega-A)^2}{\overline{q}-\underline{q}-2\omega}$).*
- *$\overline{q} - h$ thus compatibility if ($\overline{q} - \underline{q} > A + 2\omega$ and $h \geq \frac{(\overline{q}-\underline{q}-2\omega-A)^2}{\overline{q}-\underline{q}-2\omega}$) or ($\overline{q} - \underline{q} \leq A + 2\omega$ and $h > 0$).*

Proof. We just remark that the inequality $\overline{q} - \underline{q} > x_h^* + 2\omega$ is equivalent to $h < \frac{(\overline{q}-\underline{q}-2\omega-A)^2}{\overline{q}-\underline{q}-2\omega}$ when $\overline{q} - \underline{q} > A + 2\omega$. ■

4.3 Calculation of h

In this section, we calculate the optimal choice of h for the incumbent. As in the determination of the quality equilibrium, the conclusion depends on the width of the quality segment. We first study the case $\overline{q} - \underline{q} > A + 2\omega$, which allows the activity of both firms under incompatibility (Proposition 7). Then we study the case $\overline{q} - \underline{q} \leq A + 2\omega$, which prevents the activity of the entrant under incompatibility (Proposition 8). The proofs are given at the end of this section. The different equilibrium configurations:

[5] Recall that $x_{h=0}^* = A$

compatibility, incompatibility or no entry depend on the values of α and $\overline{q} - \underline{q}$ as depicted in Figure 1.[6]

Proposition 7 *If $\overline{q} - \underline{q} > A + 2\omega$ then:*

- *If $\alpha \geq \frac{\overline{\theta}-2\underline{\theta}}{3}$ the incumbent chooses $h = 0$, the entrant produces $\underline{q}$ and incompatibility prevails at equilibrium.*
- *If $0 < \alpha < \frac{\overline{\theta}-2\underline{\theta}}{3}$ then there exists $\tilde{x}(\alpha) > A$ such that :*
  - *If $\overline{q} - \underline{q} \in]A + 2\omega, \tilde{x}(\alpha) + 2\omega]$, the incumbent chooses $h = \overline{q} - \underline{q}$, the entrant produces $\underline{q}$ and compatibility prevails at equilibrium.*
  - *If $\overline{q} - \underline{q} > \tilde{x}(\alpha) + 2\omega$, the incumbent chooses $h = 0$, the entrant produces $\underline{q}$ and incompatibility prevails at equilibrium.*
- *if $\alpha = 0$ then the incumbent chooses $h = \overline{q} - \underline{q}$, the entrant produces $\underline{q}$ and compatibility prevails at equilibrium.*

Proposition 8 *If $\overline{q} - \underline{q} \leq A + 2\omega$ then*

- *If $\alpha \geq \frac{\overline{\theta}-2\underline{\theta}}{3}$, the incumbent chooses $h = 0$ and there is no entry.*
- *If $\alpha < \frac{\overline{\theta}-2\underline{\theta}}{3}$ then*
  - *If $\overline{q} - \underline{q} \in]\frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2-\underline{\theta}(\overline{\theta}-\underline{\theta})-\alpha^2}, A + 2\omega]$ the incumbent chooses $h = \overline{q} - \underline{q}$. The entrant produces $\underline{q}$ and compatibility prevails at equilibrium.*
  - *If $\overline{q} - \underline{q} \leq \frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2-\underline{\theta}(\overline{\theta}-\underline{\theta})-\alpha^2}$ the incumbent chooses $h = 0$ and there is no entry.*

Note that when there is entry the entrant always produces $\underline{q}$ as both her profit under compatibility and her profit under incompatibility are decreasing in q_e. As in a standard vertical differentiation model, at equilibrium, the product differentiation between the firms is maximal.

Figure 1 depicts the equilibrium configurations in the plane $(\overline{q} - \underline{q}, \alpha)$ and for fixed values of $\overline{\theta}$ and $\underline{\theta}$. The graphic is divided in three regions corresponding to different types of equilibria. In region I, there is no entry. Only the incumbent is active. In region II, both firms are active and incompatible at equilibrium. Finally, in region III, both firms are active and compatible.

[6] In appendix 3, we give all the details to plot the graphic of Figure 1.

Compatibility allows firms to have a large network. The willingness to pay of consumers for compatible products is higher and the equilibrium prices under compatibility are higher for both firms. However, the incumbent must support costs to achieve compatibility and the demand for the incumbent product is greater under incompatibility. A trade-off must be done between the price effect on the one hand and the compatibility cost effect and the demand effect on the other hand.

From Figure 1, we deduce that the quality segment must be sufficiently large to allow the activity of two firms under incompatibility (Region I).

When the quality segment is small ($\overline{q} - \underline{q} \leq A + 2\omega$), the incumbent can block entry choosing $h = 0$ (the entrant can only be incompatible). However, the incumbent accommodates entry for low values of α choosing $h = \overline{q} - \underline{q}$. In fact, if the incumbent chooses $h = 0$, she must set a low price to block entry ($p_I = \underline{\theta}(\overline{q} - \underline{q}) + \omega(\overline{\theta} - \underline{\theta})$) and she has all the market. If the incumbent chooses compatibility, she sets a higher price ($p_c^I = \frac{2\overline{\theta}-\underline{\theta}}{3}(\overline{q} - \underline{q})$) but renouces to a part of the market to the entrant. When α is low, the price effect dominates the demand effect.

When the quality segment is large ($\overline{q} - \underline{q} > A + 2\omega$) both firms are active at equilibrium and the same reasoning as in the case of a small quality segment holds. A trade-off is needed between the price effect and the demand and the compatibility costs effects.

Note finally that when α is very low and the quality segment large the positive price effect always outweighs the demand and compatibility costs effects. Indeed a large quality segment ($\rightarrow +\infty$) pushes up the compatibility costs $\alpha^2(\overline{q} - \underline{q})$. To choose compatibility α must be very low. This explains why the curve Γ goes to 0 as the quality segment goes to infinity.

Proof of Proposition 7. Let us examine the incumbent's profit as a function of h. We denote by $\tilde{h} = \frac{(\overline{q}-\underline{q}-2\omega-A)^2}{\overline{q}-\underline{q}-2\omega}$. As we showed previously, the entrant chooses incompatibility with $q_e = \underline{q}$ if $h < \tilde{h}$. If $h \geq \tilde{h}$, the entrant chooses compatibility with $q_e = \overline{q} - h$ (Proposition 6'). The incumbent's profit is then:

$$\Pi^I(h) = \begin{cases} (\frac{2\overline{\theta}-\underline{\theta}}{3} + \frac{\omega(\overline{\theta}+\underline{\theta})}{3(\overline{q}-\underline{q}-2\omega)})^2(\overline{q} - \underline{q} - 2\omega) - \alpha^2 h & \text{if} \quad h < \tilde{h} \\ ((\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \alpha^2)h & \text{if} \quad h \geq \tilde{h} \end{cases}$$

The incumbent's profit is always decreasing on $[0, \tilde{h}[$ and discontinuous at $\tilde{h}$. Depending on the comparison between α and $\frac{2\overline{\theta}-\underline{\theta}}{3}$, it may be decreasing or increasing on $[\tilde{h}, \overline{q} - \underline{q}]$.
When $\alpha \geq \frac{2\overline{\theta}-\underline{\theta}}{3}$. The incumbent's profit is maximal at $h = 0$.
When $\alpha < \frac{2\overline{\theta}-\underline{\theta}}{3}$, the incumbent's profit reaches its maximum at $h = 0$ or $h = \overline{q} - \underline{q}$.
$\frac{\Pi^I(0)}{\Pi^I(\overline{q}-\underline{q})} = \frac{(\frac{2\overline{\theta}-\underline{\theta}}{3}+\frac{\omega(\overline{\theta}+\underline{\theta})}{3(\overline{q}-\underline{q}-2\omega)})^2(\overline{q}-\underline{q}-2\omega)}{((\frac{2\overline{\theta}-\underline{\theta}}{3})^2-\alpha^2)(\overline{q}-\underline{q})}$ must thus be compared to 1.
Let $x = \overline{q}-\underline{q}-2\omega$ and $g(x) = \frac{(\frac{2\overline{\theta}-\underline{\theta}}{3}+\frac{\omega(\overline{\theta}+\underline{\theta})}{3x})^2 x}{((\frac{2\overline{\theta}-\underline{\theta}}{3})^2-\alpha^2)(x+2\omega)} = \frac{1}{1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}}(1+\frac{B}{x})^2\frac{x}{x+2\omega}$ with $B = \frac{\omega(\overline{\theta}+\underline{\theta})}{2\overline{\theta}-\underline{\theta}}$.

The variation table of $g(x)$ is the following:

x	A		$+\infty$
$g'(x)$		$+$	
$g(x)$	$g(A)$	$\nearrow$	$\frac{1}{1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}}$

From the variation table of $g(x)$, we deduce that if $\alpha = 0$, $\lim\limits_{x\to+\infty} g(x) = 1$ the incumbent always prefers compatibility.
If $\alpha > 0$, $\lim\limits_{x\to+\infty} g(x) > 1$ we distinguish two cases:

- If $g(A) \geq 1$ then $g(x) > 1$ for every $x > A$ and the incumbent chooses $h = 0$.
- If $g(A) < 1$ then there exists $\tilde{x} \in]A, +\infty[$ such that $g(\tilde{x}) = 1$. If $x > \tilde{x}$ then the incumbent chooses $h = 0$ and if $x \leq \tilde{x}$ the incumbent chooses $h = \overline{q} - \underline{q}$.

We have $g(A) = \frac{1}{1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}} \frac{3(\overline{\theta}^2-\underline{\theta}^2)}{(2\overline{\theta}-\underline{\theta})^2}$. $g(A) \geq 1$ is equivalent to $\alpha \geq \frac{\overline{\theta}-2\underline{\theta}}{3}$. Thus if $\alpha \geq \frac{\overline{\theta}-2\underline{\theta}}{3}$ the incumbent chooses $h = 0$. If $\alpha < \frac{\overline{\theta}-2\underline{\theta}}{3}$ there exists $\tilde{x} \in]A, +\infty[$ such that $g(\tilde{x}) = 1$. If $x > \tilde{x}$ then the incumbent chooses $h = 0$ and if $x \leq \tilde{x}$ the incumbent chooses $h = \overline{q} - \underline{q}$. $\blacksquare$

Proof of Proposition 8. Using Proposition 6', the incumbent's profit is as follows:
$$\Pi^I = \begin{cases} ((\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \alpha^2)h & \text{if} \quad h > 0 \\ (\underline{\theta}(\overline{q}-\underline{q}) + \omega(\overline{\theta}-\underline{\theta}))(\overline{\theta}-\underline{\theta}) & \text{if} \quad h = 0 \end{cases}$$
It's clear that when $\alpha \geq \frac{2\overline{\theta}-\underline{\theta}}{3}$ the optimal choice for the incumbent is $h = 0$ as her profit under compatibility is negative. Only the incumbent is active in this case.
When $\alpha < \frac{2\overline{\theta}-\underline{\theta}}{3}$, the profit is increasing on $[0, \overline{q} - \underline{q}[$. We must then compare the incumbent profit at $h = 0$ and at $h = \overline{q} - \underline{q}$. In fact $\Pi^I(h = \overline{q} - \underline{q}) > \Pi^I(h = 0)$ if and only if $\overline{q} - \underline{q} > \frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta}) - \alpha^2}$.
As $\overline{q} - \underline{q} \leq A + 2\omega$ this inequality can be satisfied if and only if $\frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta}) - \alpha^2} < A + 2\omega$ which is equivalent to $\alpha < \frac{\overline{\theta}-2\underline{\theta}}{3}$. $\blacksquare$

5 Conclusion

We analyzed in this paper the subgame perfect equilibrium of a model of vertical differentiation in the presence of network externalities. An incumbent firm offers the highest quality and chooses the compatibility interval which involves costs increasing with the width of the chosen interval. An entrant chooses which quality to produce and is thus compatible or incompatible with the incumbent depending on the quality she chooses.

Due to the network effects, the incumbent can block entry by choosing an empty compatibility interval when the quality segment is small enough, which is not possible in a standard model of vertical differentiation.

Even if the incumbent completely supports the compatibility costs and even if she can block entry, the incumbent favors in some cases compatibility. Choosing compatibility implies opposite effects on the incumbent's profit. In fact, the incumbent benefits from the positive price effect of compatibility. But, the demand for his quality is lower and she must support costs of achieving compatibility.

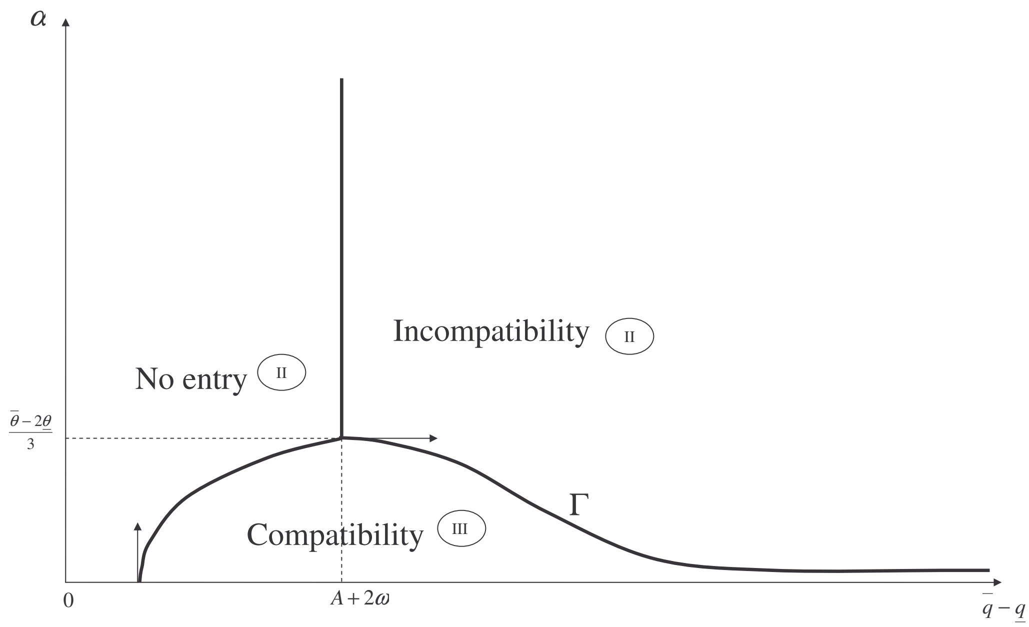

Figure 1: Equilibrium configurations

Appendix 1

We suppose now that consumers select a type 2 Nash equilibrium when a selection is needed i.e if $\overline{q} - q_e - 2\omega = 0$ and $p_I = \psi(p_e) = \varphi(p_e)$ or if $\overline{q} - q_e - 2\omega < 0$ and $p_I \in]\psi(p_e), \varphi(p_e)[$.

If $\overline{q} - q_e - 2\omega = 0$, we find that a price equilibrium does not exist if consumers select a type 2 Nash equilibrium.

If $\overline{q} - q_e - 2\omega < 0$, we find that a price equilibrium exists only if $\overline{q} - q_e \leq \omega\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}}$.

Proposition 9 *If $\overline{q} - q_e - 2\omega = 0$ and if a type 2 NE is selected (when necessary), there is no price equilibrium under incompatibility.*

Proof. If consumers select a type 2 NE then the profit of firms I and e are given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_I < p_e + \omega(\overline{\theta} + \underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_e \leq p_I - \omega(\overline{\theta} + \underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

The best reply correspondence of firm I never exists. Thus no price equilibrium exists.

■

Proposition 10 *If $\overline{q} - \underline{q} - 2\omega < 0$, and if a type 2 NE is selected (when necessary) then:*

- *If $\omega\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}} < \overline{q} - q_e < 2\omega$, there is no price equilibrium.*
- *If $\overline{q} - q_e \leq \omega\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}}$ a price equilibrium with only firm e active exists:*
  $\begin{cases} p^I_{inc} = 0 \\ p^e_{inc} = \omega(\overline{\theta} - \underline{\theta}) - \overline{\theta}(\overline{q} - q_e) \end{cases}$ $\begin{cases} \Pi^I_{inc} = 0 \\ \Pi^e_{inc} = (\omega(\overline{\theta} - \underline{\theta}) - \overline{\theta}(\overline{q} - q_e))(\overline{\theta} - \underline{\theta}) \end{cases}$

Proof. If consumers select a type 2 NE then, by Proposition 2, the profit of firms I and e are given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_I < \psi(p_e) \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_e \leq p_I - \overline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) \\ 0 & \text{otherwise} \end{cases}$$

- If $\overline{\theta}(\overline{q} - q_e) > \omega(\overline{\theta} - \underline{\theta})$, the best reply correspondence of firm I never exists. Thus no price equilibrium exists.
- If $\overline{\theta}(\overline{q} - q_e) \leq \omega(\overline{\theta} - \underline{\theta})$ then the best reply function of firm I is:
  $R^I(p_e) = \begin{cases} [0,y] & \text{if} \quad p_e \leq \omega(\overline{\theta} - \underline{\theta}) - \overline{\theta}(\overline{q} - q_e) \\ \text{doesn't exist} & \text{if} \quad p_e > \omega(\overline{\theta} - \underline{\theta}) - \overline{\theta}(\overline{q} - q_e) \end{cases}$
  The best reply function of firm e is $R^e(p_I) = p_I - \overline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta})$ for every $p_I \in [0, y]$. The unique possible intersection between the best reply correspondences is $(p_I = 0, p_e = \omega(\overline{\theta} - \underline{\theta}) - \overline{\theta}(\overline{q} - q_e))$.

■

When consumers select a type 2 Nash equilibrium (when necessary), we can have an existence problem of price equilibrium thus an existence problem of a game equilibrium..

Appendix 2

We suppose that consumers select a type 3 Nash equilibrium when necessary i.e if $\overline{q} - q_e - 2\omega < 0$ and $p_I \in]\psi(p_e), \varphi(p_e)[$. In fact, we show that there is no price equilibrium in this case. Thus no game equilibrium.

Proposition 11 *If* $\overline{q} - q_e - 2\omega < 0$ *and if a type 3 NE is selected (when necessary), there is no price equilibrium.*

Proof. By Proposition 2, we have that the profit of firms I and e are given by:

$$\Pi^I_{inc} = \begin{cases} p_I(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_I < \psi(p_e) \\ p_I(\overline{\theta} - \widehat{\theta}) & \text{if} \quad \psi(p_e) \leq p_I \leq \varphi(p_e) \\ 0 & \text{if} \quad p_I > \varphi(p_e) \end{cases}$$

$$\Pi^e_{inc} = \begin{cases} p_e(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_e < p_I - \underline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) \\ p_e(\widehat{\theta} - \underline{\theta}) & \text{if} \quad p_I - \underline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) \leq p_e \leq p_I - \overline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) \\ 0 & \text{if} \quad p_e > p_I - \overline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) \end{cases}$$

the best reply correspondence of firm I is

$$R_I(p_e) = \varphi(p_e) = p_e + \underline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) \text{ for every } p_e \in [0, y].$$

The best reply correspondence for firm e is

$$R_e(p_I) = \begin{cases} p_I - \overline{\theta}(\overline{q} - q_e) + \omega(\overline{\theta} - \underline{\theta}) & \text{if} \quad p_I \geq \overline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) \\ \text{doesn't exists} & \text{if} \quad p_I < \overline{\theta}(\overline{q} - q_e) - \omega(\overline{\theta} - \underline{\theta}) \end{cases}$$

When the best reply correspondence of firm e exists, the best reply correspondences never intersect. Thus there is no price equilibrium. ■

Appendix 3

The curve Γ in Figure 1 is divided in two parts. The first part (in region I) corresponds to $\overline{q} - \underline{q} = \frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta}) - \alpha^2}$ and is plotted for $\overline{q} - \underline{q} \leq A + 2\omega$. The second part (in region II) corresponds to $\overline{q} - \underline{q} = \tilde{x}(\alpha) + 2\omega$ and is plotted for $\overline{q} - \underline{q} > A + 2\omega$. By Propositions 7 and 8, we have to study two functions on the interval $[0, \frac{\overline{\theta}-2\underline{\theta}}{3}[$:

1. $z : \alpha \mapsto \frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta}) - \alpha^2}$
2. $\tilde{x} : \alpha \mapsto \tilde{x}(\alpha)$

1. We have that $z'(\alpha) = 2\alpha \frac{\omega(\overline{\theta}-\underline{\theta})^2}{((\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta}) - \alpha^2)^2}$; $z(0) = \frac{\omega(\overline{\theta}-\underline{\theta})^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2 - \underline{\theta}(\overline{\theta}-\underline{\theta})} > 0$ and $\lim_{\alpha \to \frac{\overline{\theta}-2\underline{\theta}}{3}} z(\alpha) = A + 2\omega$. We deduce then the variation table of $z(.)$:

$$\begin{array}{c|ccc} \alpha & 0 & & \frac{\overline{\theta}-2\underline{\theta}}{3} \\ \hline z'(\alpha) & 0 & + & \\ \hline & & & A + 2\omega \\ z(\alpha) & & \nearrow & \\ & z(0) & & \end{array}$$

From the variation table of $z(.)$ we plot the first part of the curve Γ corresponding to $\overline{q} - \underline{q} = z(\alpha)$. At $\alpha = 0$ the first part of the curve Γ has a vertical tangent in the plane $(\overline{q}-\underline{q},\alpha)$ as $z'(0)=0$.

2. We now study $\tilde{x}(\alpha)$. For $\alpha < \frac{\overline{\theta}-2\underline{\theta}}{3}$, we have that $\tilde{x}(\alpha)$ satisfies the equation:

$$g(\tilde{x}(\alpha)) = 1 \text{ with } \tilde{x}(\alpha) > A$$

$g(.)$ was defined in the proof of Proposition 7 by $g(x) = \frac{1}{1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}}(1+\frac{B}{x})^2\frac{x}{x+2\omega}$ with $x = \overline{q}-\underline{q}-2\omega$ and $B = \frac{\omega(\overline{\theta}+\underline{\theta})}{2\overline{\theta}-\underline{\theta}}$. We have that

- $g(A) = 1$ when $\alpha = \frac{\overline{\theta}-2\underline{\theta}}{3}$ thus $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}} \tilde{x}(\alpha) = A$.
- $\lim\limits_{x\to+\infty} g(x) = 1$ when $\alpha = 0$ thus $\lim\limits_{\alpha\to 0}\tilde{x}(\alpha) = +\infty$

Let us study the sign of $\tilde{x}'(\alpha)$. We deduce from the function $g(.)$ that $\tilde{x}(\alpha)$ satisfies the equality

$$(1+\frac{B}{\tilde{x}(\alpha)})^2\frac{\tilde{x}(\alpha)}{\tilde{x}(\alpha)+2\omega} = 1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2} \text{ with } \tilde{x}(\alpha) > A$$

Denote by $f_1(\alpha) = (1+\frac{B}{\tilde{x}(\alpha)})^2\frac{\tilde{x}(\alpha)}{\tilde{x}(\alpha)+2\omega}$ and $f_2(\alpha) = 1-\frac{\alpha^2}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}$.
We have that $f_1(\alpha) = f_2(\alpha)$ for every $\alpha \in]0,\frac{\overline{\theta}-2\underline{\theta}}{3}[$ thus $f_1'(\alpha) = f_2'(\alpha)$ for every $\alpha \in]0,\frac{\overline{\theta}-2\underline{\theta}}{3}[$.
From simple calculations, we have

- $f_1'(\alpha) = \frac{2(\omega-B)\tilde{x}'(\alpha)}{\tilde{x}(\alpha)(\tilde{x}(\alpha)+2\omega)^2}(1+\frac{B}{\tilde{x}(\alpha)})(\tilde{x}(\alpha)-A)$
- $f_2'(\alpha) = \frac{-2\alpha}{(\frac{2\overline{\theta}-\underline{\theta}}{3})^2}$

We thus deduce that $\tilde{x}'(\alpha) < 0$ for $\alpha \in]0,\frac{\overline{\theta}-2\underline{\theta}}{3}[$.
We now calculate $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}^-} \tilde{x}'(\alpha)$.
As $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}^-} f_2'(\alpha) < 0$, we have $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}^-} f_1'(\alpha) < 0$.
From the expression of $f_1'(\alpha)$ and as $\tilde{x}(\frac{\overline{\theta}-2\underline{\theta}}{3}) = A$, we deduce that $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}^-} \tilde{x}'(\alpha) = -\infty$
(Otherwise $\lim\limits_{\alpha\to\frac{\overline{\theta}-2\underline{\theta}}{3}^-} f_1'(\alpha) = 0$ which is not true).
We can now give the variation table of $\tilde{x}(\alpha)$.

α	0		$\frac{\overline{\theta}-2\underline{\theta}}{3}$
$\tilde{x}'(\alpha)$		$+$	$-\infty$
$\tilde{x}(\alpha)$	$+\infty$	$\searrow$	A

From the variation table of $\tilde{x}(.)$, we plot the second part of the curve Γ corresponding to $\overline{q} - \underline{q} = \tilde{x}(\alpha) + 2\omega$. At $\alpha = \frac{\overline{\theta}-2\theta}{3}$ the second part of the curve Γ has a horizontal tangent in the plane $(\overline{q} - \underline{q}, \alpha)$ as $\lim_{\alpha \to \frac{\overline{\theta}-2\theta}{3}^-} x'(\alpha) = -\infty$.

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